

Tunneling dynamics of correlated bosons in a double well potential

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Abstract. The quantum dynamics of a few bosons in a double well potential is studied using a Bose Hubbard model. We consider both signs for the on-site interparticle interaction and also investigated the situations where they are large and small. Interesting distinctive features are noted for the tunneling oscillations of these bosons corresponding to the above scenarios. Further, the sensitivity of the particle dynamics to the initial conditions has been studied. It is found that corresponding to an odd number of particles, such as three (or five), an initial condition of having unequal number of particles in the wells has interesting consequences, which is most discernible when the population difference between the wells is unity.

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1 Introduction

Ever since the ultracold bosonic atoms could be experimentally prepared by confining in a single quantum state, called a BEC [1,2], the researchers in the field have intensely tried to broaden their research on cold atoms. Manipulation of the atomic gas to emulate various quantum many body phenomena has emerged into an exciting research endeavor of the atomic physics and the condensed matter physics community which is aptly complimented by the discovery of optical lattices with precisely tunable interaction potentials via manipulating the laser parameters and Feshbach resonance. Such effects are enormously facilitated by an ultra-clean, phonon-free system, and finally, a vastly magnified version of the crystal lattice.

Tunneling of particles through a classically impenetrable barrier is a classic problem of quantum physics [3]. When interaction between these particles are included, it may help or hinder the tunneling phenomena. The time resolved tunneling probability may demonstrate interesting effects of the roles of the interaction parameter(s) and the initial configuration of the particles. For example, the phenomena of time evolved pair tunneling of the particles, as opposed to the individual tunneling across the barrier, can crucially depend upon the initial state of the system.

In the regime where the atoms are weakly interacting, tunneling phenomena of individual particles dominate, as it is the case for normal Josephson junctions. However as the (repulsive) interaction grows stronger between the

atoms, two of them located at one side of the barrier cannot tunnel independently and thus a pair tunneling becomes inevitable. There can also be a ‘conditional tunneling regime’ where tunneling of a single particle can happen only in the presence of a second particle that acts as switch [4].

A simplified version to study correlated particle dynamics in presence of confining potential is to consider a single particle or a few particles in a double well potential. Albeit straightforward, it has the potential to demonstrate a range of fundamental quantum phenomena with regard to the tunneling dynamics of the particles and abilities to manipulate them in terms of suppression of the tunneling probabilities, and thereby trapping them in one of the wells. Such trapping phenomena are experimentally realized with a BEC [5,6]. The studies involving a few bosons have turned out to be more relevant in recent times after experimental successes of the ‘Boson Sampling’ [7,8,9] where a small number of bosons were used for experimental demonstration of achieving unprecedented control of multiphoton interferences in large interferometers. These techniques offer huge prospects of simplifying the quantum computation problem and speeding it up further.

A two mode approximation, valid when the energy difference between the two lowest single particle eigenstates is far smaller than all other energy states, can describe the tunneling between different Bloch bands in an optical lattice [10]. In this work, we shall investigate a two-site Bose Hubbard model (BHM), which is the simplest candidate to investigate the dynamics of correlated bosonic atoms in a double well potential [11,12,13,14,15] or a bosonic

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junction [16]. At the outset, it is helpful to mention that we shall mainly focus on the physics of weak and strong inter-particle repulsion limits and the sensitivity of initial conditions on the tunneling dynamics.

The dynamical evolution of the Fock space for a simple two-site Bose Hubbard model (BHM) (without the density exchange term) with two bosons has been investigated and the tunneling probabilities are computed as a function of time [17,18]. However a detailed analysis of the quantum dynamics in strong and weak coupling regimes and the sensitivity of the time evolved state to a variety of initial states were lacking. This is particularly relevant for engineered waveguide lattices to achieve a certain preferred final state. Thus it is interesting to consider a few (N) bosons ($N > 2$, for example, $N = 3, 4$ etc) as the complexity of dynamics (compared to $N = 2$) is inevitable as the many body effects will become more conspicuous for a larger assembly of particles. Furthermore, an inter-site density exchange term in BHM, relevant for a gas of dipolar bosonic atoms, can also be considered in the present context. Inclusion of this term in the BHM is known to have density ordering effects and is responsible for a rich phase diagram in a lattice [19].

Motivated by the above prospects, we have considered a few bosons in a Bose Hubbard model (BHM) and investigated the quantum dynamics. In particular, we have investigated the dependencies of the tunneling probabilities in the strong and weak on-site interaction limits, briefly the effect of the density ordering term therein and the sensitivity of the dynamics to a variety of initial conditions. Among other results, the time evolved dynamics is seen to be crucially dependent on the initial state of the system in which it is prepared, particularly when the initial population difference between the two wells is unity for an odd number of particles. Further, we include a brief discussion on the effect of using an admixture of initial states on the tunneling oscillations.

In the following, the presentation of the paper is organized as follows. The next section deals with studying the quantum dynamics exactly for a system consisting of a few bosons confined in a double well potential and described by a BHM on a two site lattice. Hence, we present our results on the effect of different initial conditions on the tunneling dynamics. In particular, we have included a brief discussion on using different admixture of states as initial conditions. The implications of our results on the experimental scenario is presented thereafter.

2 The Bose-Hubbard Model and the tunneling dynamics for a few bosons

Even though we are going to restrict ourselves to the usual (short ranged) Bose Hubbard model [17,20], we include an extended density ordering term while deriving the equations of motion (EOM) with the motivation of investigating its competing effects with the on-site term on the tunneling dynamics. It is relevant to mention that for dipolar bosons, such extended range interaction potentials are

important to include, as the research of ultracold dipolar gases gained interest with the experimental realization of Bose condensed Cr atoms which hosts large long range interactions [21]. As will be immediately clear, the extended term, unlike that for a lattice, only renormalizes the on-site interaction for a double well.

For a system of N interacting bosons occupying the weakly coupled low lying energy states of a symmetric double well potential, the BHM Hamiltonian is written as,

$$\hat{H}_{BHM} = -J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{2}(\hat{a}_1^{\dagger 2} \hat{a}_1^2 + \hat{a}_2^{\dagger 2} \hat{a}_2^2) + V(\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2) = \hat{H}_J + \hat{H}_U + \hat{H}_V, \quad (1)$$

where $\hat{a}_1^\dagger(\hat{a}_2)$ are the creation (annihilation) operators of bosons in the left (right) wells, J being the tunneling parameter ($J > 0$) between the two modes, U is the strength of the on-site interaction ($U > 0$) and V is the strength of the extended density interaction (or exchange interaction) that has, as mentioned earlier, implications in formation of density order phases and are suitable in the context of dipolar bosons. It may be noted here that all the energy scales including the time evolution are expressed in units of tunneling frequency, J . It may be noted that the \hat{H}_V term includes \hat{H}_U in the following way [22],

$$\hat{H}_V = V \left[N^2 - N - \frac{\hat{H}_U}{U} \right], \quad (2)$$

where $N = n_1 + n_2 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$. In the light of this, the density exchange term (\hat{H}_V) has no role, apart from normalizing the interaction strength U to $U' (= U - V)$. This consequently implies a noninteracting scenario for $U = V$. The role of the inter-particle interaction becomes only relevant for $U \neq V$. Further, the negative sign in the expression of U' means that it can either be negative (attractive) for $U < V$ [23] or positive (repulsive) for $U > V$, for which we have considered $V = 2U$ and $V = 0.5U$ respectively as the representative values. However due to the symmetric nature of the model, a sign change in interaction does not contribute in its dynamics. Albeit, in our analysis we have used different values corresponding to attractive and repulsive region (as a pathological case) which effectively provides two different interaction strengths manifesting solely the role of interaction magnitude instead of its character. Further, we have distinguished the weak and strong coupling regimes by assuming $U = 0.1$ and $U = 12$ (both in units of the tunneling frequency, J) respectively. The ‘strong’ and ‘weak’ will carry these values along throughout the manuscript. Other representative values have been assumed for the computation of tunneling dynamics, however they yield no new qualitative inference.

To obtain the tunneling dynamics, the state vector of the system is expanded in the basis of Fock states for a constant particle number N , as in the following,

$$|\Psi(t)\rangle = \sum_{l=0}^N \frac{c_l(t)}{\sqrt{l!(N-l)!}} \hat{a}_1^{\dagger l} \hat{a}_2^{\dagger N-l} |0\rangle, \quad (3)$$

where out of N particles, l are in the left well and $N-l$ are in the right well and $c_l(t)$ s are the complex coefficients. The EOM can be written as,

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = \hat{H}_{BHM}|\Psi(t)\rangle. \quad (4)$$

In terms of the coefficients $c_l(t)$, EOM is expressed as,

$$i \frac{dc_l(t)}{dt} = -k_l c_{l+1} - k_{l-1} c_{l-1} + a_l c_l + b_l c_l, \quad (5)$$

where $k_l = J\sqrt{(l+1)(N-l)}$, $a_l = \frac{U}{2}[l^2 + (N-l)^2 - N]$, $b_l = V[l(N-l)]$

For the case two bosons ($N = 2$), Eq.(5) reduces to three coupled equations as in the following,

$$i \frac{d}{dt} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} U & -\sqrt{2}J & 0 \\ -\sqrt{2}J & V & -\sqrt{2}J \\ 0 & -\sqrt{2}J & U \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}. \quad (6)$$

Hence the particle occupation probabilities can be obtained by solving these coupled equations. The various initial conditions that can be thought of are,

$$\begin{aligned} c_0(0) = 1, c_1(0) = c_2(0) = 0; \quad c_1(0) = 1, c_0(0) = c_2(0) = 0; \\ c_2(0) = 1, c_0(0) = c_1(0) = 0. \end{aligned} \quad (7)$$

In short we shall denote them as (100), (010) and (001) respectively, where (100) means that, initially all the bosons are in the right well with the left one being empty. Similarly, (010) denotes one in each well, while (001) implies both in the left well with the right well being empty.

Similarly, for the case of three bosons, one gets four coupled equations, which are,

$$i \frac{d}{dt} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3U & -\sqrt{3}J & 0 & 0 \\ -\sqrt{3}J & U+2V & -2J & 0 \\ 0 & -2J & U+2V & -\sqrt{3}J \\ 0 & 0 & -\sqrt{3}J & 3U \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}. \quad (8)$$

As earlier, these equations can be solved for four different initial conditions, namely, (1000), (0100), (0010) and (0001), with implications as before. For example, (0100) denotes a situation where two particles are in the right well, with the other in the left well and so on.

A straightforward extension yields similar set of equations (now a set of five) and the corresponding initial conditions for $N = 4$. We have repeated the procedure till $N = 16$. For brevity, we skip them here.

An extension of our results to the case of N bosons is possible via the method of induction. A straightforward application of this method on Eqs.(6) and (8) yields,

$$i \frac{dc_0}{dt} = -\sqrt{N}Jc_1 + \frac{N(N-1)}{2}Uc_0, \quad (9)$$

where all the bosons are in the right well. Similarly the case corresponding to $(N-1)$ bosons in the right well and

one in the left, can be denoted by,

$$\begin{aligned} i \frac{dc_1}{dt} &= -\sqrt{N}Jc_0 - \sqrt{2(N-1)}Jc_2 \\ &+ \frac{N^2-3N+2}{2}Uc_1 + (N-1)Vc_1. \end{aligned}$$

For an equal distribution of the bosons with $N/2$ in each well (N : even) is

$$\begin{aligned} i \frac{dc_{N/2}}{dt} &= -\frac{J}{2}\sqrt{N(N+2)}(c_{N/2+1} + c_{N/2-1}) \\ &+ \frac{U}{4}N(N-2)c_{N/2} + \frac{V}{4}N^2c_{N/2}. \end{aligned}$$

However for discussing our results in the following section, we restrict ourselves to the case of a few bosons.

3 Physical Observables and Results

As emphasized earlier, we are interested in studying the quantum tunneling dynamics of a few bosons in a double well potential. We are mainly interested in the effect of the inter-particle repulsion and density exchange for two, three, four and five bosons and the variation in the tunneling dynamics associated with different initial conditions. In the following, we describe the cases corresponding to two, three and four bosons separately. In this regard, an useful (and experimentally measurable) quantity to study the tunneling dynamics of bosons can be the population in one of the wells (say the right well), $P_R(t)$ as discussed in the following discussion [20].

For a system of two bosons, the right well population, $P_R(t)$ can be expressed as [17],

$$P_R(t) = |c_0(t)|^2 + \frac{1}{2}|c_1(t)|^2, \quad (10)$$

which is a superposition of the probabilities of both the bosons in the right well and half of that corresponding to one in each well.

It may be noted that for $U' = 0$ (*i.e.* $U = V$), Rabi oscillation of the particles between the wells is observed (Fig.1(a)). At small values of U' , the atoms can still tunnel independently, similar to that of the normal Josephson junction, however the time period for oscillation becomes enormously large which signals the onset of a trapping scenario. A similar scenario has been reported by Zöllner *et al.* [20], where they have found that for $g = 1.3$ (g being the strength of the pairwise potential), the time period is as large as 2×10^3 s. In this situation, two or more atoms residing in one of the wells, form a 'repulsively bound pair' [24] and hence tunnel together. Such phenomena are difficult to contemplate in crystal lattices owing to relatively much shorter life times associated with the decay processes.

For the weakly interacting case $U = 0.1$, it is important to note that $P_R(t)$ collapses, as evident from Fig.1(b). However there is again a 'revival' as time progresses and

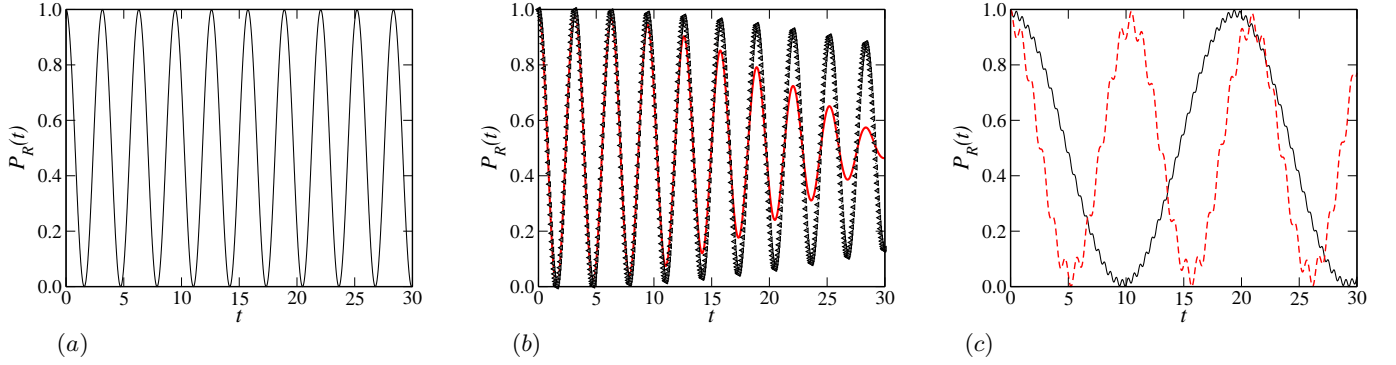


Fig. 1. (Color online) $P_R(t)$, defined in text, as a function of time for two bosons is shown for non interacting case ($V = U$) in (a). In (b) left triangles with dots (black) denote $P_R(t)$ corresponding to $U' = 0.05$ and solid line (red) denotes $P_R(t)$ corresponding to $U' = -0.1$. Similarly in (c) dashed lines (red) denote $P_R(t)$ corresponding to $U' = 6$ and solid line (black) denotes $P_R(t)$ corresponding to $U' = -12$. The time t in the x -axis is measured in units of the tunneling frequency, J and is true for all plots.

this phenomenon is repeated with increasing time. At time, $t = 0$, the system is prepared in a definite state (as described by the initial conditions in the preceding section) and the two terms in Eq.(10) are correlated. However as time increases, the oscillations corresponding to different initial excitations pick up different frequencies and hence become uncorrelated, thereby leading to a collapse. With further increase in time, the correlation is partly (depending on the value of U) restored and revival occurs. This behavior repeats itself and thus an infinite sequence of collapse and revivals are obtained [25]. At large values of U , namely, $U = 12$, termed as the *fermionization limit* (where bosons avoid each other and thus obey an ‘*exclusion principle*’), there are faster oscillations with smaller amplitudes, however $P_R(t)$ becomes zero eventually (see Fig.1(c)), signaling a tunneling of the atoms at large time scales. The time period of such ‘eventual tunneling’ phenomena, T_R (say) increases with the increase in the inter-particle repulsion, U , thereby signaling intense trapping effects. The analytic expressions for T_R corresponding to the noninteracting ($U = V$), and interacting (considering two pathological cases $V = 2U$ and $V = 0.5U$) cases are obtained as,

$$\begin{aligned} T_R &= \pi/J \quad \text{for } V = U \\ &= \left| \frac{8\pi}{U - \sqrt{64J^2 + U^2}} \right| \quad \text{for } V = 0.5U \\ &= \left| \frac{4\pi}{U - \sqrt{16J^2 + U^2}} \right| \quad \text{for } V = 2U \end{aligned} \quad (11)$$

Thus, as a function of U , T_R scales linearly and the agreement between the analytic expressions and the corresponding numeric estimates are shown in Fig. (2). So T_R is insensitive to U values for the non-interacting case, while it sharply increases as U is increased for the interacting cases.

Let us now analyze the impact of different initial conditions on the tunneling dynamics of two particles in a double well potential. To prepare an initial state with a population imbalance, a tilt in the form of a linear potential $-\eta x$ ($\eta > 0$) can be superimposed [20,26]. For a

reasonably large η (magnitude of the tilt), all the particles can be made to reside in one well. The subsequent dynamics can be studied by allowing $\eta \rightarrow 0$ within some characteristic time scale. Motivated by such prospects of experimentally creating different initial states [27], we study the tunneling dynamics subject to different initial conditions.

In the weak coupling regime ($U = 0.1$) and the so called attractive limit ($U' < 0$ or $V = 2U$), $P_R(t)$ oscillates and slowly dampens for the initial conditions given by (100) and (001) (see Fig.3(a)), while the damping is faster with further weakening of the interaction field ($U' > 0$ or $V = 0.5U$) (Fig.3(b)). The situation in the strong coupling limit ($U = 12$) show Rabi oscillations with different frequencies corresponding to the $V = 2U$ (Fig.3(c)) and $V = 0.5U$ (Fig.3(d)) situations. It may also be noted that if we start with an initial condition (010) where one boson resides in each well, Eq.(3) becomes,

$$|\Psi(t)\rangle = c_1(t)\hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle \quad (12)$$

It can be shown that the above state is an eigenstate of the Hamiltonian (Eq.(1)) and hence the dynamics is frozen which is seen from Fig.(3), where $P_R(t)$ stays at 0.5 irrespective of the values of parameters used. The frequency and time period of these oscillations depend upon the interaction parameters used in this work.

Let us now concentrate on the case of three bosons ($N = 3$). The $P_R(t)$ in this case is defined as,

$$P_R(t) = |c_0(t)|^2 + \frac{2}{3}|c_1(t)|^2 + \frac{1}{3}|c_2(t)|^2. \quad (13)$$

Similar to the case of two bosons, here $P_R(t)$ is a combination of probabilities of all of them in the right well ($|c_0(t)|^2$), two in the right and one in the left with an amplitude $\frac{2}{3}$ and one in the right and two in the left with an amplitude $\frac{1}{3}$ respectively. While qualitatively the tunneling behavior remains unaltered as compared to two bosons, with regard to Rabi oscillations at $U' = 0$ (not shown here) and there is a temporary decay of the amplitude of oscillations (Fig.4(a)) due to the beating phenomena for $U' \neq 0$. It can also be seen that $V = 0.5U$ registers

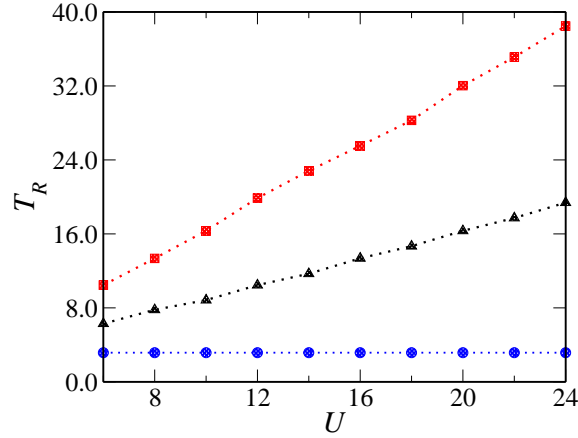


Fig. 2. (Color online) Time period T_R of two bosons as a function of U is shown. In the figure, the circles with dotted line (blue), triangles with dotted line (black) and squares with dotted line (red) denote the analytical results for the non-interacting case ($V = U$), and interacting cases with $V = 0.5U$ and $V = 2U$ respectively. The associated open squares are the numerically obtained results.

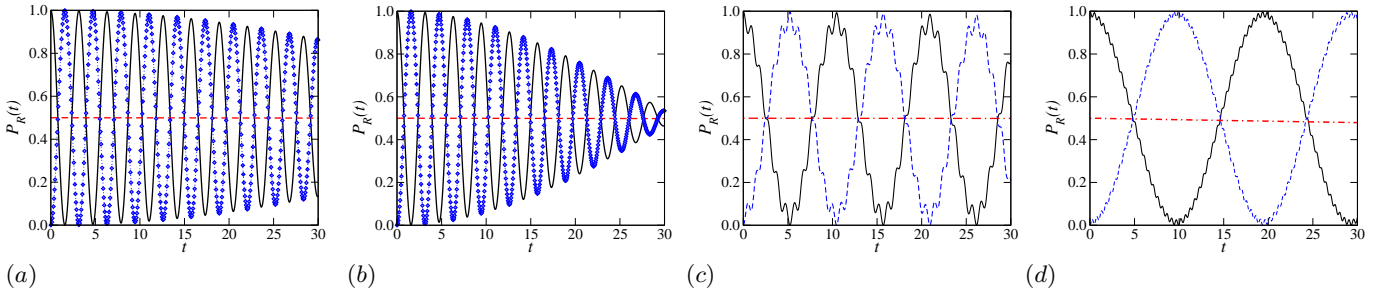


Fig. 3. (Color online) $P_R(t)$ as a function of time corresponding to three different initial conditions is shown. In (a) $U' = 0.05$, and (b) $U' = -0.1$ solid line (black) denotes (100), dashed line (red) denotes (010) and diamonds with dots (blue) denote (001). In (c) $U' = 6$ and (d) $U' = -12$ solid line (black) denotes (100), dashed-dotted line (red) denotes (010), and dashed lines (blue) denote (001). See text for details on notations.

more significant decay of the amplitude owing to trapping effects. At large U , (Fig.4(b)), the time period of ‘eventual’ oscillations becomes very large. The time period is about an order of magnitude larger compared to two bosons. In Fig.4(b), for $U' = 6$, one can observe tunneling phenomena at larger time scales, while at $U' = -12$ (large V), the tunneling of atoms take a very long time and we do not observe any tunneling until $t = 60$ and even to much large values of time (not shown here). This indicates emergence of trapping phenomena for large values of the exchange interaction V in the large U regime.

A close scrutiny of different initial conditions for three bosons in the small U regime reveals an interesting observation. $P_R(t)$, corresponding to the initial condition (0100) ((0010)), starts with $\frac{2}{3}$ ($\frac{1}{3}$), as expected. However, as time progresses, $P_R(t)$ modulates between values approximately 0.85 and 0.15 (Fig.4(c) and (d)). Thus in the weak coupling regime ($U \simeq 0.1$), the fraction of the total number of bosons occupying the right well is becoming larger (smaller) than $\frac{2}{3}$ ($\frac{1}{3}$), thereby indicating a tendency of accumulation of particles in one of the wells. This seems like an interesting result as an accumulation of particles is not expected for $U < J$ ($U = 0.1$ in units of J here). $P_R(t)$ oscillating between values such as, $\frac{2}{3}$ and $\frac{1}{3}$

may have been more commonly expected. The other initial condition, namely, (1000) (or (0001)) does not exhibit any noteworthy feature and hence not included for discussion.

Therefore, it indicates that in case of odd number of particles, when both the wells contain unequal number of particles and the population difference between the wells differs by unity, then such a scenario of accumulation of particles may be observed. However in the noninteracting limit (with $V = U$), such accumulation of particles vanishes and $P_R(t)$ oscillates between $\frac{2}{3}$ and $\frac{1}{3}$. Similar result emerges for large U limit ($U = 12$) where the accumulation of particles ceases to be a possibility owing to trapping effects. We have skipped these plots for brevity.

As an extension of the ongoing discussion, we take a look at the case of four bosons. Here $P_R(t)$ is defined as,

$$P_R(t) = |c_0(t)|^2 + \frac{3}{4}|c_1(t)|^2 + \frac{1}{2}|c_2(t)|^2 + \frac{1}{4}|c_3(t)|^2, \quad (14)$$

There is no qualitative difference in the behavior for $P_R(t)$ both in $V = 2U$ and $V = 0.5U$ cases corresponding to the weak coupling regime between this and those for two or three bosons. In the strong coupling case as expected, the localization is strong, and a complete tunneling of all the particles is prohibited over a very large time scales. Thus

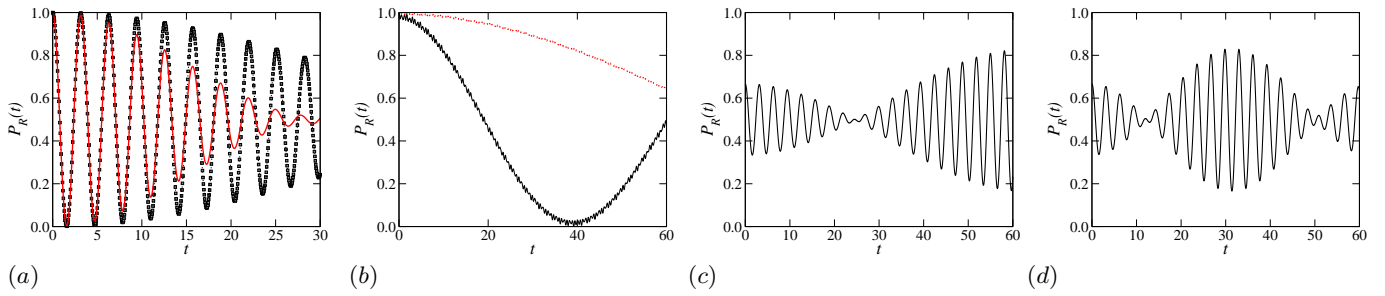


Fig. 4. (Color online) $P_R(t)$ with the initial condition (1000) for three bosons. In (a) square with dots (black) denote $U' = 0.05$ and solid line (red) denotes $U' = -0.1$ and (b) solid line (black) denotes $U' = 6$ and dotted line (red) denotes $U' = -12$. $P_R(t)$ with (0100) initial condition is depicted in (c) with $U' = 0.05$ and (d) with $U' = -0.1$.

the notion of (Rabi) oscillations at large U as inferred earlier, is no longer observed, at least for time scales $\sim 10^3$ (in limits of the tunneling frequency).

Again an inspection of $P_R(t)$ with different initial conditions such as (10000), (00001) and (00100) yield results similar to those corresponding to (100), (001) and (010), respectively, for two bosons with the last one in either case yields, $P_R(t) = 0.5$ for all t and this value is fairly insensitive to the values of U and V . For (01000) and (00010) as possible initial conditions, that is, three particle in the right well and one in the left, $P_R(t)$ oscillate between values $\frac{3}{4}$ and $\frac{1}{4}$, and vice versa as expected.

Thus the effect of initial conditions seem to be important for an odd number of bosons in a double well, specially when the population difference between the well is unity. The claim is substantiated by looking at the case of five bosons, for which, at small values of U , three particles in the right well and two in the left (or vice versa) produces probabilities nearly $\frac{4}{5}$ and $\frac{1}{5}$ as time progresses, which are greater than $\frac{3}{5}$ and $\frac{2}{5}$, thereby indicating a possibility of accumulation tendencies, where $P_R(t)$ values oscillate between $\frac{4}{5}$ and $\frac{1}{5}$ as expected. The plots are skipped here for brevity.

4 Admixture of states

Further emphasis on the effect of initial conditions can be given as follows. Instead of choosing a particular initial state, one can consider an admixture of states. For example, for the case of two particles, instead of assigning an initial state (100), we may consider an admixture of the form,

$$\beta(100) + \gamma(010) + \delta(001)$$

with the restriction, $|\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. In this spirit, we have considered small deviations from the pure states by suitably choosing β , γ and δ and looked at the time evolved states via $P_R(t)$ for a comparison with those for the pure states. The corresponding plots for some specific choices of β , γ and δ are presented in Fig.(5). A slight deviation from a pure state, say (100) as an initial state, results in a slightly different (Figs. 5 (a) and (c)) or completely different dynamics (Figs.5 (b) and (d)) where in

the latter case, a small mixing of the probability amplitudes corresponding to (010) state yields an oscillatory dynamics. Thus the probability amplitudes of the initial states can slightly be modified to yield a desired oscillatory dynamics.

5 Time averaged dynamics - extrapolation to large N

In order to draw relevance of these results to the experiments done on cold atoms, we need to extend the studies for a large number of bosons. An exact computation of the quantum dynamics for such a large system is difficult. So we present a time averaged $P_R(t)$, denoted by α , which is defined as,

$$\alpha = \frac{1}{T} \int_0^T P_R(t) dt \quad (15)$$

by computing the EOM exactly for upto 16 bosons and plotted α as a function of $1/N$ (N being the number of bosons) and hence the results are extrapolated to $N \rightarrow \infty$ (or $1/N \rightarrow 0$). Here T is taken as 30 in units of $1/J$ (J being the tunneling amplitude). The results in the weak and strong coupling limits are presented in Fig.(6). In Fig.6(a) which corresponds to a small U regime, α does not depend on N and stays at an average value of 0.5, regardless of whether interaction effects have been included. In the large U regime (in fig. Fig.6(b)), α , although flat, yet different for different values of V corresponding to smaller number of particles, shows a linear fall off as N becomes large. In the limit $N \rightarrow \infty$, α becomes small 0.1 – 0.2 (the extrapolated value), re emphasizing the onset of the trapping effects as the time averaged probability for the particles to spend in one of the wells (right well here) becomes low. Hence there is indeed a depreciation in the value of the time averaged right well population in presence of a large number of bosons in a double well potential, however qualitatively similar physics can be expected as that for a few bosons. Summarizing the above discussion, we conclude by saying that the tunneling period not only increases with the particle number, but also depends on the interaction strength. Saturation behaviour of the time period (that is intense trapping) is expected in the limit of N for the strongly interacting regime (which is clear from Fig.(6)).

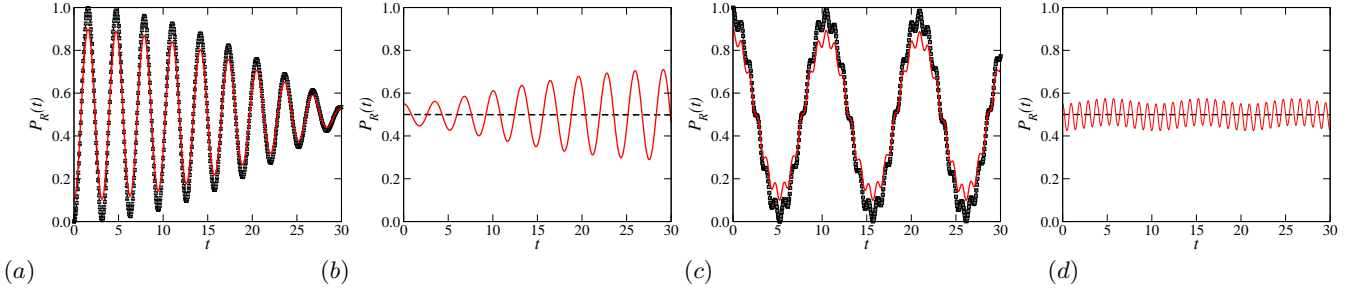


Fig. 5. (Color online) $P_R(t)$ with different admixtures of initial conditions are shown for two bosons. In (a) $U' = -0.1$, squares with dashed lines (black) denote the initial condition (100) and solid line (red) denotes the initial condition $(\sqrt{0.9}, 0, \sqrt{0.1})$, (b) $U' = -0.1$, dashed lines (black) denote the initial condition (010) and solid line (red) denotes the initial condition $(\sqrt{0.1}, \sqrt{0.9}, 0)$, (c) $U' = 6$, squares with dashed lines (black) denote the initial condition (100) and solid line (red) denotes the initial condition $(\sqrt{0.9}, 0, \sqrt{0.1})$, (d) $U' = 6$ here dashed line (black) denotes the initial condition (010) and solid line (red) denotes the initial condition $(\sqrt{0.1}, \sqrt{0.9}, 0)$. A small admixture leads to an oscillatory dynamics in (b) and (d).

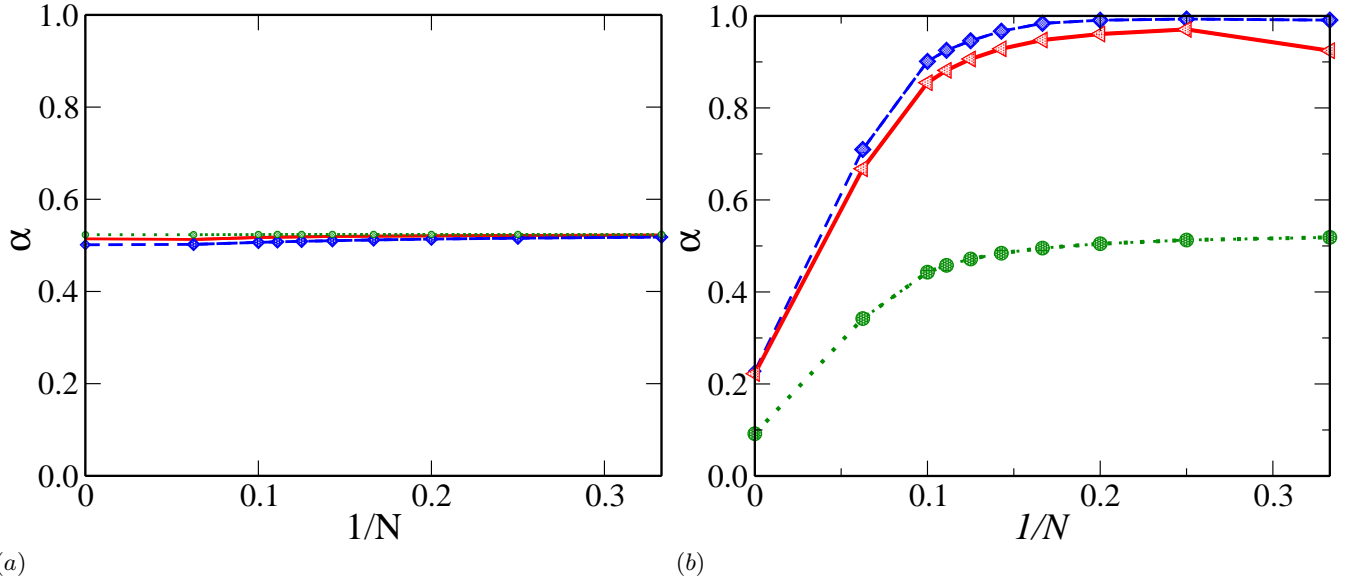


Fig. 6. (Color online) Time integrated value of $P_R(t)$ i.e. α as a function of inverse of the number of bosons is shown for (a) weak and (b) strong coupling cases. In (a) circles with dotted line (green) denotes α for $V = U$ (non-interacting), solid line (red) denotes α for $V = 0.5U$ and diamonds with dashed line denotes α for $V = 2U$. In (b) circles with dotted line (green) denotes α for $V = U$ (non-interacting), right triangles with solid line (red) denotes α for $V = 0.5U$ and diamonds with dashed line (blue) denotes α for $V = 2U$.

6 Conclusions

We have carried out a detailed enumeration, though by no means exhaustive, of the effects of onsite inter-particle repulsion on the tunneling dynamics of a few bosons in a double well potential. The strong and weak coupling limits are compared and contrasted with regard to the study of tunneling dynamics. Further, the sensitivity of the particle dynamics to different initial conditions is closely scrutinized. For an odd number of particles in the limit of weak repulsion, a population difference of one particle among the two wells seems to demonstrate accumulation tendencies. However, no such behavior is observed for the population difference to be larger than one. Also the effect of an admixture of initial states on the tunneling oscillations has been studied. It is premature to comment on the implication of this result to more elegant phenomena, such

as using it as an ‘atom switch’ etc, however we feel that our results can motivate further experiments in the study of atomic dynamics in presence of correlation effects.

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Author Contribution

All authors contributed equally.

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